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Analysis in $\mathbb{R}^{1,1}$ or the Principal Function Theory*

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Abstract

We explore a function theory connected with the principal series representation of $SL(2,\mathbb{R})$ in contrast to standard complex analysis connected with the discrete series. We construct counterparts for the Cauchy integral formula, the Hardy space, the Cauchy-Riemann equation and the Taylor expansion.

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Contents

1 Introduction

You should complete your own original research in order to learn when it was done before.

Connections between complex analysis (one variable, several complex variables, Clifford analysis) and its symmetry groups are known from its earliest days and are an obligatory part of the textbook on the subject [3], [6], [8, § 1.4, § 5.4], [19], [21, Chap. 2]. However ideas about fundamental role of symmetries in function theories outlined in [7, 18] were not incorporated in a working toolkit of researchers yet.

It was proposed in [14] to distinguish essentially different function theories by corresponding group of symmetries. Such a classification is needed because not all seemingly different function theories are essentially different [12]. But it is also important that the group approach gives a constructive way to develop essentially different function theories [14, 16], as well as outlines an alternative ground for functional calculi of operators [13]. In the mentioned papers all given examples consider only well-known function theories. While rearranging of known results is not completely useless there was an appeal to produce a new function theory based on the described scheme.

The theorem proved in [15] underlines the similarity between structure of the group of Möbius transformations in spaces $\mathbb{R}^n$ and $\mathbb{R}^{pq}$. This generates a hope that there exists a non empty function theory in $\mathbb{R}^{pq}$. We construct such a theory in the present paper for the case of $\mathbb{R}^{1,1}$. Other new function theories based on the same scheme will be described elsewhere [4].

The format of paper is as follows. In Section 2 we introduce basic notations and definitions. We construct two function theories—the standard

complex analysis and a function theory in $\mathbb{R}^{1,1}$—in Section 3. Our consideration is based on two different series of representation of $SL(2,\mathbb{R})$: discrete and principal. We deduce in their terms the Cauchy integral formula, the Hardy spaces, the Cauchy-Riemann equation, the Taylor expansion and their counterparts for $\mathbb{R}^{1,1}$. Finally we collect in Appendix A several facts, which we would like (however can not) to assume well known. It may be a good idea to look through the Appendixes A.1–A.4 between the reading of Sections 2 and 3. Finally Appendix A.6 states few among many open problems. Our examples will be rather lengthy thus their (not always obvious) ends will be indicated by the symbol $\lozenge$.

There are no claims about novelty any particular formula for analysis in $\mathbb{R}^{1,1}$. Most of them are probably known in some different form in the theory of special functions (see for example 3.11). Moreover our general ideas are very close to [7]. However the composition of results as a function theory parallel to the complex analysis is believed to be new.

2 Preliminaries

Let $\mathbb{R}^{pq}$ be a real n-dimensional vector space, where $n = p + q$ with a fixed frame $e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_n$ and with the nondegenerate bilinear form $B(\cdot,\cdot)$ of the signature (p,q), which is diagonal in the frame e_i, i.e.:

$$B(e_i, e_j) = \epsilon_i \delta_{ij}, \text{ where } \epsilon_i = \begin{cases} 1, & i = 1, \ldots, p \\ -1, & i = p+1, \ldots, n \end{cases}$$

and δ_{ij} is the Kronecker delta. In particular the usual Euclidean space $\mathbb{R}^n$ is $\mathbb{R}^{0n}$. Let $\mathcal{C}\ell(p,q)$ be the *real Clifford algebra* generated by 1, e_j, $1 \le j \le n$ and the relations

$$e_i e_j + e_j e_i = -2B(e_i, e_j).$$

We put $e_0 = 1$ also. Then there is the natural embedding $\mathfrak{i} : \mathbb{R}^{pq} \to \mathcal{C}\ell(p,q)$. We identify $\mathbb{R}^{pq}$ with its image under $\mathfrak{i}$ and call its elements *vectors*. There are two linear anti-automorphisms $*$ (reversion) and $-$ (main anti-automorphisms) and automorphism $'$ of $\mathcal{C}\ell(p,q)$ defined on its basis $A_\nu = e_{j_1} e_{j_2} \cdots e_{j_r}$, $1 \le j_1 < \cdots < j_r \le n$ by the rule:

$$(A_\nu)^* = (-1)^{\frac{r(r-1)}{2}} A_\nu, \qquad \bar{A}_\nu = (-1)^{\frac{r(r+1)}{2}} A_\nu, \qquad A'_\nu = (-1)^r A_\nu.$$

In particular, for vectors, $\bar{\mathbf{x}} = \mathbf{x}' = -\mathbf{x}$ and $\mathbf{x}^* = \mathbf{x}$.

It is easy to see that $\mathbf{x}\mathbf{y} = \mathbf{y}\mathbf{x} = 1$ for any $\mathbf{x} \in \mathbb{R}^{pq}$ such that $B(\mathbf{x},\mathbf{x}) \neq 0$ and $\mathbf{y} = \bar{\mathbf{x}} \left\|\mathbf{x}\right\|^{-2}$, which is the *Kelvin inverse* of $\mathbf{x}$. Finite products of invertible vectors are invertible in $\mathcal{C}\ell(p,q)$ and form the *Clifford group* $\Gamma(p,q)$.

Elements $a \in \Gamma(p,q)$ such that $a\bar{a} = \pm 1$ form the $\mathrm{Pin}(p,q)$ group—the double cover of the group of orthogonal rotations $\mathrm{O}(p,q)$. We also consider [3, § 5.2] $T(p,q)$ to be the set of all products of vectors in $\mathbb{R}^{pq}$.

Let (a,b,c,d) be a quadruple from $T(p,q)$ with the properties:

1. $(ad^* - bc^*) \in \mathbb{R} \setminus 0$;
2. a^*b, c^*d, ac^*, bd^* are vectors.

Then [3, Theorem 5.2.3] 2×2-matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form the group $\Gamma(p+1,q+1)$ under the usual matrix multiplication. It has a representation $\pi_{\mathbb{R}^{pq}}$ by transformations of $\overline{\mathbb{R}^{pq}}$ given by:

$$\pi_{\mathbb{R}^{pq}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{x} \mapsto (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1}, \tag{2.1}$$

which form the *Möbius* (or the *conformal*) group of $\overline{\mathbb{R}^{pq}}$. Here $\overline{\mathbb{R}^{pq}}$ the compactification of $\mathbb{R}^{pq}$ by the "necessary number of points" (which form the light cone) at infinity (see [3, § 5.1]). The analogy with fractional-linear transformations of the complex line $\mathbb{C}$ is useful, as well as representations of shifts $\mathbf{x} \mapsto \mathbf{x} + y$, orthogonal rotations $\mathbf{x} \mapsto k(a)\mathbf{x}$, dilations $\mathbf{x} \mapsto \lambda\mathbf{x}$, and the Kelvin inverse $\mathbf{x} \mapsto \mathbf{x}^{-1}$ by the matrixes $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$, $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively. We also use the agreement of [3] that $\frac{a}{b}$ always denotes ab^{-1} for a, $b \in \mathcal{C}\ell(p,q)$.

3 Two Function Theories Associated with representations of $SL(2,\mathbb{R})$

3.1 Unit Disks in $\mathbb{R}^{0,2}$ and $\mathbb{R}^{1,1}$

We describe a coherent states and wavelet transform connected with a homogeneous space $\Omega = G/H$ and unitary irreducible representation π of G, which is induced by a character of H [11, § 13.2]. This representation is assumed to be a square integrable with respect to Ω (see below).

Let G be a locally compact group and H be its closed subgroup. Let $\Omega = G/H$ and $s : \Omega \to G$ be a continuous mapping [11, § 13.2]. Then any $g \in G$ has a unique decomposition of the form $g = s(\omega)h$, $\omega \in \Omega$ and we will write $\omega = s^{-1}(g)$, $h = r(g) = \left(s^{-1}(g)\right)^{-1} g$. Note that Ω is a

left G-homogeneous space with an action defined in terms of s as follow: $g : a \mapsto g \cdot \omega = s^{-1}(g^{-1} * s(\omega))$, where $*$ is the multiplication on G.

The main example is provided by group $G = SL(2,\mathbb{R})$ (books [10, 20, 23] are our standard references about $SL(2,\mathbb{R})$ and its representations) consisting of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries and determinant $ad - bc = 1$. See Appendix A.1 for description of its Lie algebra. We will construct two series of examples. One is connected with discreet series representation and produces the core of standard complex analysis. The second will be its mirror in principal series representations and create parallel function theory. $SL(2,\mathbb{R})$ has also other type representation, which can be of particular interest in other circumstances. However the discreet series and principal ones stay separately from others (in particular by being the support of the Plancherel measure [20, § VIII.4], [23, Chap. 8, (4.16)]) and are in a good resemblance each other.

Example 3.1.(a) Via identities

$$\alpha = \frac{1}{2}(a + d - ic + ib), \qquad \beta = \frac{1}{2}(c + b - ia + id)$$

we have isomorphism of $SL(2,\mathbb{R})$ with group $SU(1,1)$ of 2×2 matrices with complex entries of the form $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ such that $|\alpha|^2 - |\beta|^2 = 1$. We will use the last form for $SL(2,\mathbb{R})$ for complex analysis in unit disk $\mathbb{D}$.

$SL(2,\mathbb{R})$ has the only non-trivial compact closed subgroup K, namely the group of matrices of the form $h_\psi = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$. Now any $g \in SL(2,\mathbb{R})$ has a unique decomposition of the form

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} &= |\alpha| \begin{pmatrix} 1 & \beta\bar\alpha^{-1} \\ \bar\beta\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\bar\alpha}{|\alpha|} \end{pmatrix} \\ &= \frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} 1 & a \\ \bar a & 1 \end{pmatrix} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \end{aligned} \tag{3.1}$$

where $\psi = \Im \ln(\alpha)$, $a = \beta\bar\alpha^{-1}$, and $|a| < 1$ because $|\alpha|^2 - |\beta|^2 = 1$. Thus we can identify $SL(2,\mathbb{R})/H$ with the unit disk $\mathbb{D}$ and define mapping $s : \mathbb{D} \rightarrow SL(2,\mathbb{R})$ as follows

$$s : a \mapsto \frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} 1 & a \\ \bar a & 1 \end{pmatrix}. \tag{3.2}$$

Mapping $r : G \to H$ associated to s is

$$r : \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\bar{\alpha}}{|\alpha|} \end{pmatrix} \tag{3.3}$$

The invariant measure $d\mu(a)$ on $\mathbb{D}$ coming from decomposition $dg = d\mu(a)\, dk$, where dg and dk are Haar measures on G and K respectively, is equal to

$$d\mu(a) = \frac{da}{\left(1 - |a|^2\right)^2}. \tag{3.4}$$

with da—the standard Lebesgue measure on $\mathbb{D}$.

The formula $g : a \mapsto g \cdot a = s^{-1}(g^{-1} * s(a))$ associates with a matrix $g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ the fraction-linear transformation of $\mathbb{D}$ of the form

$$g : z \mapsto g \cdot z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \qquad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{3.5}$$

which also can be considered as a transformation of $\dot{\mathbb{C}}$ (the one-point compactification of $\mathbb{C}$). $\Diamond$

Example 3.1.(b) We will describe a version of previous formulas corresponding to geometry of unit disk in $\mathbb{R}^{1,1}$. For generators e_1 and e_2 of $\mathbb{R}^{1,1}$ (here $e_1^2 = -e_2^2 = -1$.) we see that matrices $\begin{pmatrix} a & be_2 \\ ce_2 & d \end{pmatrix}$ again give a realization of $SL(2,\mathbb{R})$. Making composition with the Caley transform

$$T = \frac{1}{2}\begin{pmatrix} 1 & e_2 \\ e_2 & -1 \end{pmatrix}\begin{pmatrix} 1 & e_1 \\ e_1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + e_2 e_1 & e_1 + e_2 \\ e_2 - e_1 & e_2 e_1 - 1 \end{pmatrix}$$

and its inverse

$$T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -e_1 \\ -e_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & e_2 \\ e_2 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 - e_1 e_2 & e_2 + e_1 \\ e_2 - e_1 & -1 - e_1 e_2 \end{pmatrix}$$

(see analogous calculation in [20, § IX.1]) we obtain another realization of $SL(2,\mathbb{R})$:

$$\frac{1}{4}\begin{pmatrix} 1 - e_1 e_2 & e_2 + e_1 \\ e_2 - e_1 & -1 - e_1 e_2 \end{pmatrix}\begin{pmatrix} a & be_2 \\ ce_2 & d \end{pmatrix}\begin{pmatrix} 1 + e_2 e_1 & e_1 + e_2 \\ e_2 - e_1 & e_2 e_1 - 1 \end{pmatrix} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{b}' & \mathsf{a}' \end{pmatrix}, \tag{3.6}$$

where

$$\mathsf{a} = \frac{1}{2}(a(1 - e_1e_2) + d(1 + e_1e_2)), \qquad \mathsf{b} = \frac{1}{2}(b(e_1 - e_2) + c(e_1 + e_2)). \quad (3.7)$$

It is easy to check that the condition $ad - bc = 1$ implies the following value of the pseudodeterminant of the matrix $\mathsf{a}(\mathsf{a}')^* - \mathsf{b}(\mathsf{b}')^* = \mathsf{a}\bar{\mathsf{a}} - \mathsf{b}\bar{\mathsf{b}} = 1$. We also observe that a is an even Clifford number and b is a vector thus $\mathsf{a}' = \mathsf{a}$, $\mathsf{b}' = -\mathsf{b}$.

Now we consider the decomposition

$$\begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix} = |\mathsf{a}| \begin{pmatrix} 1 & \mathsf{b}\mathsf{a}^{-1} \\ -\mathsf{b}\mathsf{a}^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{\mathsf{a}}{|\mathsf{a}|} & 0 \\ 0 & \frac{\mathsf{a}}{|\mathsf{a}|} \end{pmatrix}. \quad (3.8)$$

It is seen directly, or alternatively follows from general characterization of $\Gamma(p+1, q+1)$ [3, Theorem 5.2.3(b)], that $\mathsf{b}\mathsf{a}^{-1} \in \mathbb{R}^{1,1}$. Note that now we cannot derive from $\mathsf{a}\bar{\mathsf{a}} - \mathsf{b}\bar{\mathsf{b}} = 1$ that $\mathsf{b}\mathsf{a}^{-1}\overline{\mathsf{b}\mathsf{a}^{-1}} = -(\mathsf{b}\mathsf{a}^{-1})^2 < 1$ because $\mathsf{a}\bar{\mathsf{a}}$ can be positive or negative (but we are sure that $(\mathsf{b}\mathsf{a}^{-1})^2 \neq -1$). For this reason we cannot define the unit disk in $\mathbb{R}^{1,1}$ by the condition $|\mathsf{u}| < 1$ in a way consistent with its Möbius transformations. This topic will be discussed elsewhere in more details and illustrations [5]. We describe a ready solution in Appendix A.2.

Matrices of the form

$$\begin{pmatrix} \mathsf{a} & 0 \\ 0 & \mathsf{a}' \end{pmatrix} = \begin{pmatrix} e^{e_1e_2\tau} & 0 \\ 0 & e^{e_1e_2\tau} \end{pmatrix}, \qquad \mathsf{a} = e^{e_1e_2\tau} = \cosh\tau + e_1e_2\sinh\tau, \quad \tau \in \mathbb{R}$$

comprise a subgroup of $SL(2,\mathbb{R})$ which we denote by A. This subgroup is an image of the subgroup A in the Iwasawa decomposition $SL(2,\mathbb{R}) = ANK$ [20, § III.1] under the transformation (3.6).

We define an embedding s of $\widetilde{\mathbb{D}}$ for our realization of $SL(2,\mathbb{R})$ by the formula:

$$s : \mathsf{u} \mapsto \frac{1}{\sqrt{1+\mathsf{u}^2}} \begin{pmatrix} 1 & \mathsf{u} \\ -\mathsf{u} & 1 \end{pmatrix}. \quad (3.9)$$

The formula $g : \mathsf{u} \mapsto s^{-1}(g \cdot s(\mathsf{u}))$ associated with a matrix $g^{-1} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix}$ gives the fraction-linear transformation $\widetilde{\mathbb{D}} \to \widetilde{\mathbb{D}}$ of the form:

$$g : \mathsf{u} \mapsto g \cdot \mathsf{u} = \frac{\mathsf{a}\mathsf{u} + \mathsf{b}}{-\mathsf{b}\mathsf{u} + \mathsf{a}}, \qquad g^{-1} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix} \quad (3.10)$$

The mapping $r : G \to H$ associated to s defined in (3.9) is

$$r : \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\mathsf{a}}{|\mathsf{a}|} & 0 \\ 0 & \frac{\mathsf{a}}{|\mathsf{a}|} \end{pmatrix} \quad (3.11)$$

And finally the invariant measure on $\widetilde{\mathbb{D}}$

$$d\mu(\mathsf{u}) = \frac{d\mathsf{u}}{(1+\mathsf{u}^2)^2} = \frac{du_1 du_2}{(1-u_1^2+u_2^2)^2}. \tag{3.12}$$

follows from the elegant consideration in [3, § 6.1]. $\diamondsuit$

We hope the reader notes the explicit similarity between these two examples. Following examples will explore it further.

3.2 Reduced Wavelet Transform—the Cauchy Integral Formula

Let $\chi : H \rightarrow \mathbb{C}$ be a unitary character of H, which induces in the sense of Mackey an irreducible unitary representation π of G [11, § 13.2]. We denote by the same letter π the canonical realization of this representation in the space $L_2(\Omega)$ given by the formula [11, § 13.2.(7)–(9)]:

$$[\pi(g)f](\omega) = \chi_0(r(g^{-1} * s(\omega)))f(g\cdot\omega), \qquad \chi_0(h) = \chi(h)\left(\frac{d\mu(h\cdot\omega)}{d\mu(\omega)}\right)^{\frac{1}{2}}, \tag{3.13}$$

where $g \in G$, $\omega \in \Omega$, $h \in H$ and $r : G \rightarrow H$, $s : \Omega \rightarrow G$ are functions defined at the beginning of this Section; $*$ denotes multiplication on G and $\cdot$ denotes the action of G on Ω from the left.

Let π_0 be a representation of G by operators on a Hilbert space $L_2(X, d\nu)$ such that its restriction to a subspace $F_2(X, d\nu) \subset L_2(X, d\nu)$ is an irreducible unitary representation unitary equivalent to π. It follows in particular that

$$[\pi_0(h)f](x) = \chi_0(h)f(h\cdot x), \qquad \forall h \in H, \quad f(x) \in F_2(X, d\nu). \tag{3.14}$$

Definition 3.2 Let $f_0(x) \in F_2(X, d\nu)$ be an eigenfunction for all $\pi_0(h)$, $h \in H$, i.e, $\pi_0(h)f_0(x) = \chi_0(h)f_0(x)$. It is called the *vacuum vector*. For a fixed vacuum vector $f_0(x)$ we define the *reduced wavelet transform* [16] $\mathcal{W} : L_2(X, d\nu) \rightarrow L_\infty(\Omega)$ by the formula

$$[\mathcal{W}f](\omega) = \langle f, \pi_0(s(\omega))f_0\rangle. \tag{3.15}$$

We say that a representation π_0 is *square integrable* $\bmod H$ if the reduced wavelet transform $[\mathcal{W}f_0](\omega)$ of the vacuum vector $f_0(x)$ is square integrable on Ω.

The reduced wavelet transform has following properties.

Proposition 3.3 *The reduced wavelet transform is a continuous mapping* $\mathcal{W} : L_2(X, d\nu) \to L_\infty(\Omega)$. *The kernel of the reduced wavelet transform* $\mathcal{W}$ *is the orthogonal completion* $F_2(X)^\perp$ *of* $F_2(X)$. *If* π_0 *is square integrable* $\operatorname{mod} H$ *then*

1. $\mathcal{W}$ *maps* $L_2(X)$ *to* $L_2(\Omega)$.
2. *The image of* $\mathcal{W}$ *is the closure of linear combinations of vectors* $\widehat{f}_\omega = \mathcal{W}\pi_0(s(\omega))f_0$.
3. $\mathcal{W}$ *intertwines* π_0 *and* π:

$$\mathcal{W}\pi_0(g) = \pi(g)\mathcal{W}. \tag{3.16}$$

Proof. It is obvious from (3.15) that every function in $F_2(X)^\perp$ belongs to the kernel of $\mathcal{W}$. The intersection of the kernel with $F_2(X)$ is a subspace of $F_2(X)$ invariant under π_0. Due to non-degeneracy of π_0 and its irreducibility on $F_2(X)$ this invariant subspace is trivial.

Due to the irreducibility of π_0 on $F_2(X)$ the vacuum vector $f_0(x)$ is cyclic in $F_2(X)$. Thus linear combinations of $\pi_0(g)f_0(x)$, $g \in G$ are dense in $F_2(X)$. Due to the trivial action (3.14) of H the same is true for a smaller set $\pi_0(s(\omega))f_0$, $\omega \in \Omega$. Thus the $\mathcal{W}\pi_0(s(\omega))f_0$, $\omega \in \Omega$ are dense in the image of $\mathcal{W}$.

Finally, (3.16) follows from the direct calculation:

$$\begin{aligned}
[\mathcal{W}\pi_0(g)f](\omega) &= \langle \pi_0(g)f, \pi_0(s(\omega))f_0 \rangle \\
&= \langle f, \pi_0(g^{-1})\pi_0(s(\omega))f_0 \rangle \\
&= \langle f, \pi_0(g^{-1} * s(\omega))f_0 \rangle \\
&= \chi_0(r(g^{-1} * s(\omega)))\langle f, \pi_0(s(g \cdot \omega))f_0 \rangle \\
&= \chi_0(r(g^{-1} * s(\omega)))[\mathcal{W}f](g \cdot \omega) \\
&= \pi(g)[\mathcal{W}f](\omega).
\end{aligned}$$

□

Example 3.4.(a) We continue to consider the case of $G = SL(2, \mathbb{R})$ and $H = K$. The compact group $K \sim \mathbb{T}$ has a discrete set of characters $\chi_m(h_\phi) = e^{-im\phi}$, $m \in \mathbb{Z}$. We drop the trivial character χ_0 and remark that characters χ_m and χ_{-m} give similar holomorphic and *anti*holomorphic

series of representations. Thus we will consider only characters χ_m with $m = 1, 2, 3, \ldots$.

There is a difference in behavior of characters χ_1 and χ_m for $m = 2, 3, \ldots$ and we will consider them separately.

First we describe χ_1. Let us take $X = \mathbb{T}$—the unit circle equipped with the standard Lebesgue measure $d\phi$ normalized in such a way that

$$\int_{\mathbb{T}} |f_0(\phi)|^2 \, d\phi = 1 \text{ with } f_0(\phi) \equiv 1. \tag{3.17}$$

From (3.2) and (3.3) one can find that

$$r(g^{-1} * s(e^{i\phi})) = \frac{\bar{\beta} e^{i\phi} + \bar{\alpha}}{\left|\bar{\beta} e^{i\phi} + \bar{\alpha}\right|}, \qquad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Then the action of G on $\mathbb{T}$ defined by (3.5), the equality $d(g \cdot \phi)/d\phi = \left|\bar{\beta} e^{i\phi} + \bar{\alpha}\right|^{-2}$ and the character χ_1 give the following realization of the formula (3.13):

$$[\pi_1(g) f](e^{i\phi}) = \frac{1}{\bar{\beta} e^{i\phi} + \bar{\alpha}} f\left(\frac{\alpha e^{i\phi} + \beta}{\bar{\beta} e^{i\phi} + \bar{\alpha}}\right). \tag{3.18}$$

This is a unitary representation—the *mock discrete series* of $SL(2, \mathbb{R})$ [23, ˜8.4]. It is easily seen that K acts in a trivial way (3.14) by multiplication by $\chi(e^{i\phi})$. The function $f_0(e^{i\phi}) \equiv 1$ mentioned in (3.17) transforms as follows

$$[\pi_1(g) f_0](e^{i\phi}) = \frac{1}{\bar{\beta} e^{i\phi} + \bar{\alpha}} \tag{3.19}$$

and in particular has an obvious property $[\pi_1(h_\psi) f_0](\phi) = e^{i\psi} f_0(\phi)$, i.e. it is a *vacuum vector* with respect to the subgroup H. The smallest linear subspace $F_2(X) \in L_2(X)$ spanned by (3.19) consists of boundary values of analytic functions in the unit disk and is the *Hardy space*. Now the reduced wavelet transform (3.15) takes the form

$$\begin{aligned} \widehat{f}(a) = [\mathcal{W} f](a) &= \langle f(x), \pi_1(s(a)) f_0(x) \rangle_{L_2(X)} \\ &= \int_{\mathbb{T}} f(e^{i\phi}) \frac{\sqrt{1 - |a|^2}}{\bar{a} e^{i\phi} - 1} \, d\phi \\ &= \frac{\sqrt{1 - |a|^2}}{i} \int_{\mathbb{T}} \frac{f(e^{i\phi})}{a - e^{i\phi}} i e^{i\phi} \, d\phi \\ &= \frac{\sqrt{1 - |a|^2}}{i} \int_{\mathbb{T}} \frac{f(z)}{a - z} \, dz, \end{aligned} \tag{3.20}$$

where $z = e^{i\phi}$. Of course (3.20) is the *Cauchy integral formula* up to factor $2\pi\sqrt{1-|a|^2}$. Thus we will write $f(a) = \left(2\pi\sqrt{1-|a|^2}\right)^{-1}\widehat{f}(a)$ for analytic extension of $f(\phi)$ to the unit disk. The factor 2π is due to our normalization (3.17) and $\sqrt{1-|a|^2}$ is connected with the invariant measure on $\mathbb{D}$.

Let us now consider characters χ_m ($m = 2, 3, \ldots$). These characters together with action (3.5) of G give following realization of (3.13):

$$[\pi_m(g)f](w) = f\left(\frac{\alpha w+\beta}{\bar\beta w+\bar\alpha}\right)(\bar\beta w+\bar\alpha)^{-m}. \tag{3.21}$$

For any integer $m \geq 2$ one can select a measure

$$d\mu_m(w) = 4^{1-m}(1-|w|^2)^{m-2}dw,$$

where dw is the standard Lebesgue measure on $\mathbb{D}$, such that (3.21) become unitary representations [20, § IX.3], [23, § 8.4]. These are discrete series.

If we again select $f_0(w) \equiv 1$ then

$$[\pi_m(g)f_0](w) = (\bar\beta w+\bar\alpha)^{-m}.$$

In particular $[\pi_m(h_\phi)f_0](w) = e^{im\phi}f_0(w)$ so this again is a vacuum vector with respect to K. The irreducible subspace $F_2(\mathbb{D})$ generated by $f_0(w)$ consists of analytic functions and is the m-th Bergman space (actually Bergman considered only $m = 2$). Now the transformation (3.15) takes the form

$$\begin{aligned}\widehat{f}(a) &= \langle f(w), [\pi_m(s(a))f_0](w)\rangle\\ &= \left(\sqrt{1-|a|^2}\right)^m\int_{\mathbb{D}}\frac{f(w)}{(a\bar w-1)^m}\frac{dw}{(1-|w|^2)^{2-m}},\end{aligned}$$

which for $m = 2$ is the classical Bergman formula up to factor $\left(\sqrt{1-|a|^2}\right)^m$. Note that calculations in standard approaches are "rather lengthy and must be done in stages" [19, § 1.4]. $\diamondsuit$

Example 3.4.(b) Now we consider the same group $G = SL(2,\mathbb{R})$ but pick up another subgroup $H = A$. Let $e_{12} := e_1e_2$. It follows from (A.4) that the mapping from the subgroup $A \sim \mathbb{R}$ to even numbers[1] $\chi_\sigma : a \mapsto a^{e_{12}\sigma} = (\exp(e_1e_2\sigma\ln a)) = (a\mathsf{p}_1 + a^{-1}\mathsf{p}_2)^\sigma$ parametrized by $\sigma \in \mathbb{R}$ is a character (in

[1] See Appendix A.3 for a definition of functions of even Clifford numbers.

a somewhat generalized sense). It represents an isometric rotation of $\widetilde{\mathbb{T}}$ by the angle a.

Under the present conditions we have from (3.9) and (3.11):

$$r(g^{-1} * s(\mathsf{u})) = \begin{pmatrix} \frac{-\mathsf{bu}+\mathsf{a}}{|-\mathsf{bu}+\mathsf{a}|} & 0 \\ 0 & \frac{-\mathsf{bu}+\mathsf{a}}{|-\mathsf{bu}+\mathsf{a}|} \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix}.$$

If we again introduce the exponential coordinates t on $\widetilde{\mathbb{T}}$ coming from the subgroup A (i.e., $\mathsf{u} = e_1 e^{e_1 e_2 t} \cosh t e_1 - \sinh t e_2 = (x + \frac{1}{x})e_1 - (x - \frac{1}{x})e_2$, $x = e^t$) then the measure dt on $\widetilde{\mathbb{T}}$ will satisfy the transformation condition

$$\frac{d(g \cdot t)}{dt} = \frac{1}{(be^{-t} + a)(ce^t + d)} = \frac{1}{(-\mathsf{bu} + \mathsf{a})(\mathsf{ub} - \mathsf{a})},$$

where

$$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix}.$$

Furthermore we can construct a realization of (3.13) on the functions defined on $\widetilde{\mathbb{T}}$ by the formula:

$$[\pi_\sigma(g) f](\mathsf{v}) = \frac{(-\mathsf{vb} + \bar{\mathsf{a}})^\sigma}{(-\mathsf{bv} + \mathsf{a})^{1+\sigma}} f\left(\frac{\mathsf{av} + \mathsf{b}}{-\mathsf{bv} + \mathsf{a}}\right), \quad g^{-1} = \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix}. \tag{3.22}$$

It is induced by the character χ_σ due to formula $-\mathsf{bv}+\mathsf{a} = (cx+d)\mathsf{p}_1 + (bx^{-1} + a)\mathsf{p}_2$, where $x = e^t$ and it is a cousin of the principal series representation (see [20, § VI.6, Theorem 8], [23, § 8.2, Theorem 2.2] and Appendix A.4). The subspaces of vector valued and even number valued functions are invariant under (3.22) and the representation is unitary with respect to the following inner product (about Clifford valued inner product see [3, § 3]):

$$\langle f_1, f_2 \rangle_{\widetilde{\mathbb{T}}} = \int_{\widetilde{\mathbb{T}}} \bar{f}_2(t) f_1(t)\, dt.$$

We will denote by $L_2(\widetilde{\mathbb{T}})$ the space of $\mathbb{R}^{1,1}$-even Clifford number valued functions on $\widetilde{\mathbb{T}}$ equipped with the above inner product.

We select function $f_0(\mathsf{u}) \equiv 1$ neglecting the fact that it does not belong to $L_2(\widetilde{\mathbb{T}})$. Its transformations

$$f_g(\mathsf{v}) = [\pi_\sigma(g) f_0](\mathsf{v}) = \left|1 + \mathsf{u}^2\right|^{1/2} \frac{(-\mathsf{vb} + \bar{\mathsf{a}})^\sigma}{(-\mathsf{bv} + \mathsf{a})^{1+\sigma}} \tag{3.23}$$

and in particular the identity $[\pi_\sigma(g)f_0](\mathsf{v}) = \bar{\mathsf{a}}^\sigma \mathsf{a}^{-1-\sigma} f_0(\mathsf{v}) = \mathsf{a}^{-1-2\sigma} f_0(\mathsf{v})$ for $g^{-1} = \begin{pmatrix} \mathsf{a} & 0 \\ 0 & \mathsf{a} \end{pmatrix}$ demonstrates that it is a vacuum vector. Thus we define the reduced wavelet transform accordingly to (3.9) and (3.15) by the formula:

$$
\begin{aligned}
[\mathcal{W}_\sigma f](\mathsf{u}) &= \int_{\widetilde{\mathbb{T}}} \left|1+\mathsf{u}^2\right|^{1/2} \overline{\left(\frac{(-e_1 e^{e_{12}t}\mathsf{u}+1)^\sigma}{(-\mathsf{u}e_1 e^{e_{12}t}+1)^{1+\sigma}} \right)} f(t)\, dt \\
&= \left|1+\mathsf{u}^2\right|^{1/2} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{(-e^{-e_{12}t}e_1\mathsf{u}+1)^{1+\sigma}} f(t)\, dt \qquad (3.24)\\
&= \left|1+\mathsf{u}^2\right|^{1/2} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{e^{-e_{12}t(1+\sigma)}(-e_1\mathsf{u}+e^{e_{12}t})^{1+\sigma}} f(t)\, dt \\
&= \left|1+\mathsf{u}^2\right|^{1/2} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{(-e_1\mathsf{u}+e^{e_{12}t})^{1+\sigma}} e^{e_{12}t(1+\sigma)} f(t)\, dt \\
&= \left|1+\mathsf{u}^2\right|^{1/2} e_{12} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{(-e_1\mathsf{u}+e^{e_{12}t})^{1+\sigma}} e^{e_{12}t\sigma} (e_{12}e^{e_{12}t}\, dt)\, f(t) \\
&= \left|1+\mathsf{u}^2\right|^{1/2} e_{12} \int_{\widetilde{\mathbb{T}}} \frac{(-\mathsf{u}e_1\mathsf{z}+1)^\sigma \mathsf{z}^\sigma}{(-e_1\mathsf{u}+\mathsf{z})^{1+\sigma}}\, d\mathsf{z}\, f(\mathsf{z}) \qquad (3.25)
\end{aligned}
$$

where $\mathsf{z} = e^{e_{12}t}$ and $d\mathsf{z} = e_{12}e^{e_{12}t}\, dt$ are the new monogenic variable and its differential respectively. The integral (3.25) is a singular one, its four singular points are intersections of the light cone with the origin in u with the unit circle $\widetilde{\mathbb{T}}$. See Appendix A.5 about the meaning of this singular integral operator.

The explicit similarity between (3.20) and (3.25) allows us consider transformation $\mathcal{W}_\sigma$ (3.25) as the analog of the Cauchy integral formula and the linear space $H_\sigma(\widetilde{\mathbb{T}})$ (A.7) generated by the coherent states $f_\mathsf{u}(\mathsf{z})$ (3.23) as the correspondence of the Hardy space. Due to "indiscrete" (i.e. they are not square integrable) nature of principal series representations there are no counterparts for the Bergman projection and Bergman space. $\Diamond$

3.3 The Dirac (Cauchy-Riemann) and Laplace Operators

Consideration of Lie groups is hardly possible without consideration of their Lie algebras, which are naturally represented by left and right invariant vectors fields on groups. On a homogeneous space $\Omega = G/H$ we have also defined a left action of G and can be interested in left invariant vector fields (first order differential operators). Due to the irreducibility of $F_2(\Omega)$ under left action of G every such vector field D restricted to $F_2(\Omega)$ is a scalar mul-

tiplier of identity $D|_{F_2(\Omega)} = cI$. We are in particular interested in the case $c = 0$.

Definition 3.5 [1, 17] A G-invariant first order differential operator

$$D_\tau : C_\infty(\Omega, \mathcal{S} \otimes V_\tau) \to C_\infty(\Omega, \mathcal{S} \otimes V_\tau)$$

such that $\mathcal{W}(F_2(X)) \subset \ker D_\tau$ is called *(Cauchy-Riemann-)Dirac operator* on $\Omega = G/H$ associated with an irreducible representation τ of H in a space V_τ and a spinor bundle $\mathcal{S}$.

The Dirac operator is explicitly defined by the formula [17, (3.1)]:

$$D_\tau = \sum_{j=1}^{n} \rho(Y_j) \otimes c(Y_j) \otimes 1, \tag{3.26}$$

where Y_j is an orthonormal basis of $\mathfrak{p} = \mathfrak{h}^\perp$—the orthogonal completion of the Lie algebra $\mathfrak{h}$ of the subgroup H in the Lie algebra $\mathfrak{g}$ of G; $\rho(Y_j)$ is the infinitesimal generator of the right action of G on Ω; $c(Y_j)$ is Clifford multiplication by $Y_i \in \mathfrak{p}$ on the Clifford module $\mathcal{S}$. We also define an invariant Laplacian by the formula

$$\Delta_\tau = \sum_{j=1}^{n} \rho(Y_j)^2 \otimes \epsilon_j \otimes 1, \tag{3.27}$$

where $\epsilon_j = c(Y_j)^2$ is $+1$ or -1.

Proposition 3.6 *Let all commutators of vectors of $\mathfrak{h}^\perp$ belong to $\mathfrak{h}$, i.e. $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subset \mathfrak{h}$. Let also f_0 be an eigenfunction for all vectors of $\mathfrak{h}$ with eigenvalue* 0 *and let also $\mathcal{W}f_0$ be a null solution to the Dirac operator D. Then $\Delta f(x) = 0$ for all $f(x) \in F_2(\Omega)$.*

PROOF. Because Δ is a linear operator and $F_2(\Omega)$ is generated by $\pi_0(s(a))\mathcal{W}f_0$ it is enough to check that $\Delta\pi_0(s(a))\mathcal{W}f_0 = 0$. Because Δ and π_0 commute it is enough to check that $\Delta\mathcal{W}f_0 = 0$. Now we observe that

$$\Delta = D^2 - \sum_{i,j} \rho([Y_i, Y_j]) \otimes c(Y_i)c(Y_j) \otimes 1.$$

Thus the desired assertion is follows from two identities: $D\mathcal{W}f_0 = 0$ and $\rho([Y_i, Y_j])\mathcal{W}f_0 = 0$, $[Y_i, Y_j] \in H$. $\square$

Example 3.7.(a) Let $G = SL(2,\mathbb{R})$ and H be its one-dimensional compact subgroup generated by an element $Z \in \mathfrak{sl}(2,\mathbb{R})$ (see Appendix A.1 for a description of $\mathfrak{sl}(2,\mathbb{R})$). Then $\mathfrak{h}^{\perp}$ is spanned by two vectors $Y_1 = A$ and $Y_2 = B$. In such a situation we can use $\mathbb{C}$ instead of the Clifford algebra $\mathcal{C}\ell(0,2)$. Then formula (3.26) takes a simple form $D = r(A + iB)$. Infinitesimal action of this operator in the upper-half plane follows from calculation in [20, VI.5(8), IX.5(3)], it is $[D_{\mathbb{H}} f](z) = -2iy\frac{\partial f(z)}{\partial \bar{z}}$, $z = x + iy$. Making the Caley transform we can find its action in the unit disk $D_{\mathbb{D}}$: again the Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$ is its principal component. We calculate $D_{\mathbb{H}}$ explicitly now to stress the similarity with $\mathbb{R}^{1,1}$ case.

For the upper half plane $\mathbb{H}$ we have following formulas:

$$\begin{aligned}
s &: \mathbb{H} \rightarrow SL(2,\mathbb{R}) : z = x + iy \mapsto g = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix};\\
s^{-1} &: SL(2,\mathbb{R}) \rightarrow \mathbb{H} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto z = \frac{ai + b}{ci + d};\\
\rho(g) &: \mathbb{H} \rightarrow \mathbb{H} : z \mapsto s^{-1}(s(z) * g)\\
&\qquad\qquad = s^{-1}\begin{pmatrix} ay^{-1/2} + cxy^{-1/2} & by^{1/2} + dxy^{-1/2} \\ cy^{-1/2} & dy^{-1/2} \end{pmatrix}\\
&\qquad\qquad = \frac{(yb + xd) + i(ay + cx)}{ci + d}
\end{aligned}$$

Thus the right action of $SL(2,\mathbb{R})$ on $\mathbb{H}$ is given by the formula

$$\rho(g)z = \frac{(yb + xd) + i(ay + cx)}{ci + d} = x + y\frac{bd + ac}{c^2 + d^2} + iy\frac{1}{c^2 + d^2}.$$

For A and B in $\mathfrak{sl}(2,\mathbb{R})$ we have:

$$\rho(e^{At})z = x + iye^{2t}, \qquad \rho(e^{Bt})z = x + y\frac{e^{2t} - e^{-2t}}{e^{2t} + e^{-2t}} + iy\frac{4}{e^{2t} + e^{-2t}}.$$

Thus

$$\begin{aligned}
[\rho(A)f](z) &= \frac{\partial f(\rho(e^{At})z)}{\partial t}|_{t=0} = 2y\partial_2 f(z),\\
[\rho(B)f](z) &= \frac{\partial f(\rho(e^{Bt})z)}{\partial t}|_{t=0} = 2y\partial_1 f(z),
\end{aligned}$$

where ∂_1 and ∂_2 are derivatives of $f(z)$ with respect to real and imaginary party of z respectively. Thus we get

$$D_{\mathbb{H}} = i\rho(A) + \rho(B) = 2yi\partial_2 + 2y\partial_1 = 2y\frac{\partial}{\partial \bar{z}}$$

as was expected. $\Diamond$

Example 3.7.(b) In $\mathbb{R}^{1,1}$ the element $B \in \mathfrak{sl}$ generates the subgroup H and its orthogonal completion is spanned by B and Z. Thus the associated Dirac operator has the form $D = e_1\rho(B) + e_2\rho(Z)$. We need infinitesimal generators of the right action ρ on the "left" half plane $\widetilde{\mathbb{H}}$. Again we have a set of formulas similar to the classic case:

$$\begin{aligned}
s &: \widetilde{\mathbb{H}} \rightarrow SL(2,\mathbb{R}) : \mathsf{z} = e_1 y + e_2 x \mapsto g = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix};\\
s^{-1} &: SL(2,\mathbb{R}) \rightarrow \widetilde{\mathbb{H}} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mathsf{z} = \frac{ae_1 + be_2}{ce_2e_1 + d};\\
\rho(g) &: \widetilde{\mathbb{H}} \rightarrow \widetilde{\mathbb{H}} : \mathsf{z} \mapsto s^{-1}(s(\mathsf{z}) * g)\\
&\qquad = s^{-1}\begin{pmatrix} ay^{-1/2} + cxy^{-1/2} & by^{1/2} + dxy^{-1/2} \\ cy^{-1/2} & dy^{-1/2} \end{pmatrix}\\
&\qquad = \frac{(yb + xd)e_2 + (ay + cx)e_1}{ce_2e_1 + d}
\end{aligned}$$

Thus the right action of $SL(2,\mathbb{R})$ on $\mathbb{H}$ is given by the formula

$$\rho(g)\mathsf{z} = \frac{(yb + xd)e_2 + (ay + cx)e_1}{ce_2e_1 + d} = e_1 y\frac{-1}{c^2 - d^2} + e_2 x + e_2 y\frac{ac - bd}{c^2 - d^2}.$$

For A and Z in $\mathfrak{sl}(2,\mathbb{R})$ we have:

$$\begin{aligned}
\rho(e^{At})\mathsf{z} &= e_1 y e^{2t} + e_2 x,\\
\rho(e^{Zt})\mathsf{z} &= e_1 y\frac{-1}{\sin^2 t - \cos^2 t} + e_2 y\frac{-2\sin t\cos t}{\sin^2 t - \cos^2 t} + e_2 x\\
&= e_1 y\frac{1}{\cos 2t} + e_2 y\tan 2t + e_2 x.
\end{aligned}$$

Thus

$$\begin{aligned}
[\rho(A)f](\mathsf{z}) &= \frac{\partial f(\rho(e^{At})\mathsf{z})}{\partial t}|_{t=0} = 2y\partial_2 f(\mathsf{z}),\\
[\rho(Z)f](\mathsf{z}) &= \frac{\partial f(\rho(e^{Zt})\mathsf{z})}{\partial t}|_{t=0} = 2y\partial_1 f(\mathsf{z}),
\end{aligned}$$

where ∂_1 and ∂_2 are derivatives of $f(\mathsf{z})$ with respect of e_1 and e_2 components of z respectively. Thus we get

$$D_{\widetilde{\mathbb{H}}} = e_1\rho(Z) + e_2\rho(A) = 2y(e_1\partial_1 + e_2\partial_2).$$

In this case the Dirac operator is not elliptic and as a consequence we have in particular a singular Cauchy integral formula (3.25). Another manifestation

of the same property is that primitives in the "Taylor expansion" do not belong to $F_2(\widetilde{\mathbb{T}})$ itself (see Example 3.10.(b)). It is known that solutions of a hyperbolic system (unlike the elliptic one) essentially depend on the behavior of the boundary value data. Thus function theory in $\mathbb{R}^{1,1}$ is not defined only by the hyperbolic Dirac equation alone but also by an appropriate boundary condition. $\diamondsuit$

3.4 The Taylor expansion

For any decomposition $f_a(x) = \sum_\alpha \psi_\alpha(x)V_\alpha(a)$ of the coherent states $f_a(x)$ by means of functions $V_\alpha(a)$ (where the sum can become eventually an integral) we have the *Taylor expansion*

$$\begin{aligned}
\widehat{f}(a) &= \int_X f(x)\bar{f}_a(x)\,dx = \int_X f(x)\sum_\alpha \bar{\psi}_\alpha(x)\bar{V}_\alpha(a)\,dx \\
&= \sum_\alpha \int_X f(x)\bar{\psi}_\alpha(x)\,dx\bar{V}_\alpha(a) \\
&= \sum_\alpha^\infty \bar{V}_\alpha(a) f_\alpha,
\end{aligned} \tag{3.28}$$

where $f_\alpha = \int_X f(x)\bar{\psi}_\alpha(x)\,dx$. However to be useful within the presented scheme such a decomposition should be connected with the structures of G, H, and the representation π_0. We will use a decomposition of $f_a(x)$ by the eigenfunctions of the operators $\pi_0(h)$, $h \in \mathfrak{h}$.

Definition 3.8 Let $F_2 = \int_A H_\alpha\,d\alpha$ be a spectral decomposition with respect to the operators $\pi_0(h)$, $h \in \mathfrak{h}$. Then the decomposition

$$f_a(x) = \int_A V_\alpha(a) f_\alpha(x)\,d\alpha, \tag{3.29}$$

where $f_\alpha(x) \in H_\alpha$ and $V_\alpha(a) : H_\alpha \to H_\alpha$ is called the Taylor decomposition of the Cauchy kernel $f_a(x)$.

Note that the Dirac operator D is defined in the terms of left invariant shifts and therefor commutes with all $\pi_0(h)$. Thus it also has a spectral decomposition over spectral subspaces of $\pi_0(h)$:

$$D = \int_A D_\delta\,d\delta. \tag{3.30}$$

We have obvious property

Proposition 3.9 *If spectral measures* $d\alpha$ *and* $d\delta$ *from* (3.29) *and* (3.30) *have disjoint supports then the image of the Cauchy integral belongs to the kernel of the Dirac operator.*

For discrete series representation functions $f_\alpha(x)$ can be found in F_2 (as in Example 3.9.(a)), for the principal series representation this is not the case. To overcome confusion one can think about the Fourier transform on the real line. It can be regarded as a continuous decomposition of a function $f(x) \in L_2(\mathbb{R})$ over a set of harmonics $e^{i\xi x}$ neither of those belongs to $L_2(\mathbb{R})$. This has a lot of common with the Example 3.10.(b).

Example 3.10.(a) Let $G = SL(2,\mathbb{R})$ and $H = K$ be its maximal compact subgroup and π_1 be described by (3.18). H acts on $\mathbb{T}$ by rotations. It is one dimensional and eigenfunctions of its generator Z are parametrized by integers (due to compactness of K). Moreover, on the irreducible Hardy space these are positive integers $n = 1, 2, 3\ldots$ and corresponding eigenfunctions are $f_n(\phi) = e^{i(n-1)\phi}$. Negative integers span the space of anti-holomorphic function and the splitting reflects the existence of analytic structure given by the Cauchy-Riemann equation. The decomposition of coherent states $f_a(\phi)$ by means of this functions is well known:

$$f_a(\phi) = \frac{\sqrt{1-|a|^2}}{\bar{a}e^{i\phi}-1} = \sum_{n=1}^{\infty}\sqrt{1-|a|^2}\,\bar{a}^{n-1}e^{i(n-1)\phi} = \sum_{n=1}^{\infty}V_n(a)f_n(\phi),$$

where $V_n(a) = \sqrt{1-|a|^2}\,\bar{a}^{n-1}$. This is the classical Taylor expansion up to multipliers coming from the invariant measure. $\lozenge$

Example 3.10.(b) Let $G = SL(2,\mathbb{R})$, $H = A$, and π_σ be described by (3.22). Subgroup H acts on $\widetilde{\mathbb{T}}$ by hyperbolic rotations:

$$\tau : \mathsf{z} = e_1 e^{e_{12}t} \rightarrow e^{2e_{12}\tau}\mathsf{z} = e_1 e^{e_{12}(2\tau+t)}, \qquad t, \tau \in \widetilde{\mathbb{T}}.$$

Then for every $p \in \mathbb{R}$ the function $f_p(\mathsf{z}) = (\mathsf{z})^p = e^{e_{12}pt}$ where $\mathsf{z} = e^{e_{12}t}$ is an eigenfunction in the representation (3.22) for a generator a of the subgroup $H = A$ with the eigenvalue $2(p-\sigma)-1$. Again, due to the analytical structure reflected in the Dirac operator, we only need negative values of p to decompose the Cauchy integral kernel.

Proposition 3.11 *For* $\sigma = 0$ *the Cauchy integral kernel* (3.25) *has the following decomposition:*

$$\frac{1}{-e_1\mathsf{u}+\mathsf{z}} = \int_0^\infty \frac{(e_1\mathsf{u})^{[p]}-1}{e_1\mathsf{u}-1}\cdot t\mathsf{z}^{-p}\,dp, \tag{3.31}$$

where $\mathsf{u} = u_1 e_1 + u_2 e_2$, $\mathsf{z} = e^{e_{12} t}$, *and* $[p]$ *is the integer part of* p *(i.e.* $k = [p] \in \mathbb{Z}$, $k \le p < k+1$*).*

PROOF. Let

$$f(t) = \int_0^\infty F(p) e^{-tp}\, dp$$

be the Laplace transform. We use the formula [2, Laplace Transform Table, p. 479, (66)]

$$\frac{1}{t(e^{kt} - a)} = \int_0^\infty \frac{a^{[p/k]} - 1}{a - 1} e^{-tp}\, dp \tag{3.32}$$

with the particular value of the parameter $k = 1$. Then using $\mathsf{p}_{1,2}$ defined in (A.3) we have

$$\begin{aligned}
&\int_0^\infty \frac{(e_1\mathsf{u})^{[p]} - 1}{e_1 \mathsf{u} - 1} \cdot t\mathsf{z}^{-p}\, dp = \\
&= t \int_0^\infty \left(\frac{(-u_1 - u_2)^{[p]} - 1}{(-u_1 - u_2) - 1} \mathsf{p}_2 + \frac{(-u_1 + u_2)^{[p]} - 1}{(-u_1 + u_2) - 1} \mathsf{p}_1 \right) (e^{tp} \mathsf{p}_2 + e^{-tp} \mathsf{p}_1)\, dp \\
&= t \int_0^\infty \frac{(-u_1 - u_2)^{[p]} - 1}{(-u_1 - u_2) - 1} e^{tp}\, dp\, \mathsf{p}_2 + t \int_0^\infty \frac{(-u_1 + u_2)^{[p]} - 1}{(-u_1 + u_2) - 1} e^{-tp}\, dp\, \mathsf{p}_1 \\
&= \frac{t}{t(e^{-t} + u_1 + u_2)} \mathsf{p}_2 + \frac{t}{t(e^{t} + u_1 - u_2)} \mathsf{p}_1 \qquad (3.33) \\
&= \frac{1}{(e^{-t} + u_1 + u_2)\mathsf{p}_2 + (e^{t} + u_1 - u_2)\mathsf{p}_1} \\
&= \frac{1}{-e_1 \mathsf{u} + \mathsf{z}},
\end{aligned}$$

where we obtain (3.33) by an application of (3.32). □

Thereafter for a function $f(\mathsf{z}) \in F_2(\widetilde{\mathbb{T}})$ we have the following Taylor expansion of its wavelet transform:

$$[\mathcal{W}_0 f](u) = \int_0^\infty \frac{(e_1\mathsf{u})^{[p]} - 1}{e_1 \mathsf{u} - 1} f_p\, dp,$$

where

$$f_p = \int_{\widetilde{\mathbb{T}}} t\mathsf{z}^{-p}\, d\mathsf{z} f(\mathsf{z}).$$

The last integral is similar to the Mellin transform [20, § III.3], [23, Chap. 8, (3.12)], which naturally arises in study of the principal series representations of $SL(2, \mathbb{R})$.

I was pointed by Dr. J. Cnops that for the Cauchy kernel $(-e_1\mathsf{u}+\mathsf{z})$ there is still a decomposition of the form $(-e_1\mathsf{u}+\mathsf{z})=\sum_{j=0}^{\infty}(e_1\mathsf{u})^j\mathsf{z}^{-j-1}$. It this connection one may note that representations π_1 (3.18) and π_σ (3.22) for $\sigma=0$ are unitary equivalent. (this is a meeting point between discrete and principal series). Thus a function theory in $\mathbb{R}^{1,1}$ with the value $\sigma=0$ could carry many properties known from the complex analysis. $\diamondsuit$

A Appendix

A.1 (The Lie algebra of $SL(2,\mathbb{R})$) The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ consists of all 2×2 real matrices of trace zero. One can introduce a basis

$$A=\frac{1}{2}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad B=\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad Z=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

The commutator relations are

$$[Z,A]=2B,\qquad [Z,B]=-2A,\qquad [A,B]=-\frac{1}{2}Z.$$

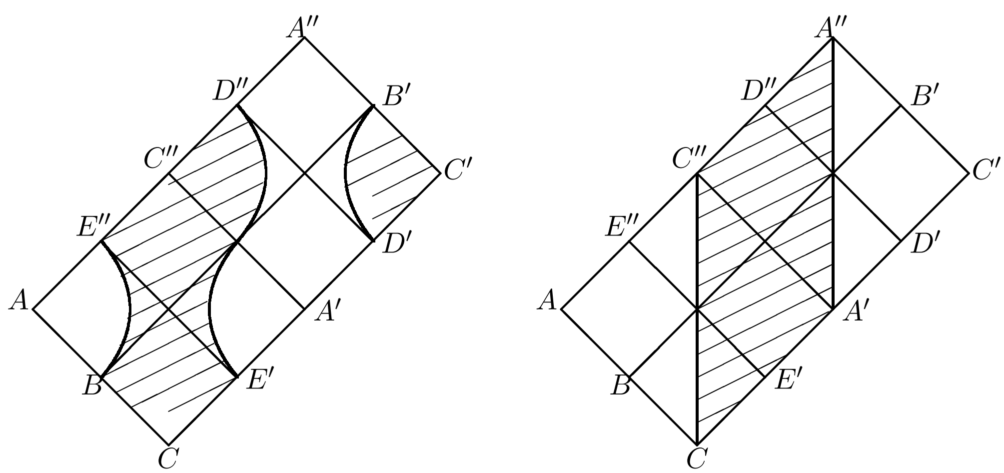

Figure 1: The conformal unit disk, unit circle (on the left) and “left” half plane (on the right) in $\mathbb{R}^{1,1}$. Points labeled by the same letters should be glued together (on each picture separately). Squares $ACA'C''$ and $A'C'A''C''$ represent $\mathbb{R}^{1,1}_-$ and $\mathbb{R}^{1,1}_+$ respectively. Their boundary are the image of the light cone at infinity.

A.2 (Conformal unit disk, unit circle, and half plane) We are taking two copies $\mathbb{R}^{1,1}_{+}$ and $\mathbb{R}^{1,1}_{-}$ of $\mathbb{R}^{1,1}$ glued over their light cones at infinity in such a way that the construction is invariant under natural action of the Möbius transformation. This aggregate denoted by $\widetilde{\mathbb{R}}^{1,1}$ is a two-fold cover of $\mathbb{R}^{1,1}$. Topologically $\widetilde{\mathbb{R}}^{1,1}$ is equivalent to the Klein bottle. Similar conformally invariant two-fold cover of the Minkowski space-time was constructed in [22, § III.4] in connection with the red shift problem in extragalactic astronomy.

We define *(conformal) unit disk* in $\widetilde{\mathbb{R}}^{1,1}$ as follows:

$$\widetilde{\mathbb{D}} = \{\mathsf{u} \mid \mathsf{u}^2 < -1,\ \mathsf{u} \in \mathbb{R}^{1,1}_{+}\} \cup \{\mathsf{u} \mid \mathsf{u}^2 > -1,\ \mathsf{u} \in \mathbb{R}^{1,1}_{-}\}. \tag{A.1}$$

It can be shown that $\widetilde{\mathbb{D}}$ is conformally invariant and has a boundary $\widetilde{\mathbb{T}}$—the two glued copies of unit circles in $\mathbb{R}^{1,1}_{+}$ and $\mathbb{R}^{1,1}_{-}$.

We call $\widetilde{\mathbb{T}}$ the *(conformal) unit circle* in $\mathbb{R}^{1,1}$. $\widetilde{\mathbb{T}}$ consists of four parts—branches of hyperbola—with subgroup $A \in SL(2,\mathbb{R})$ acting simply transitively on each of them. Thus we will regard $\widetilde{\mathbb{T}}$ as $\mathbb{R} \cup \mathbb{R} \cup \mathbb{R} \cup \mathbb{R}$ with an exponential mapping $\exp : t \mapsto (+\text{or}-)e_1^{+\text{or}-}$, $e_1^{\pm} \in \mathbb{R}^{1,1}_{\pm}$, where each of four possible sign combinations is realized on a particular copy of $\mathbb{R}$. More generally we define a set of concentric circles for $-1 \leq \lambda < 0$:

$$\widetilde{\mathbb{T}}^{\lambda} = \{\mathsf{u} \mid \mathsf{u}^2 = -\lambda^2,\ \mathsf{u} \in \mathbb{R}^{1,1}_{+}\} \cup \{\mathsf{u} \mid \mathsf{u}^2 = -\lambda^{-2},\ \mathsf{u} \in \mathbb{R}^{1,1}_{-}\}. \tag{A.2}$$

Figure 1 illustrates geometry of the conformal unit disk in $\widetilde{\mathbb{R}}^{1,1}$ as well as the "left" half plane conformally equivalent to it.

A.3 (Functions of even Clifford numbers) Let

$$\mathsf{a} = a_1\mathsf{p}_1 + a_2\mathsf{p}_2, \qquad \mathsf{p}_1 = \frac{1+e_1e_2}{2}, \quad \mathsf{p}_2 = \frac{1-e_1e_2}{2}, \quad a_1, a_2 \in \mathbb{R} \tag{A.3}$$

be an even Clifford number in $\mathcal{C}\!\ell(1,1)$. It follows from the identities

$$\mathsf{p}_1\mathsf{p}_2 = \mathsf{p}_2\mathsf{p}_1 = 0, \qquad \mathsf{p}_1^2 = \mathsf{p}_1, \qquad \mathsf{p}_2^2 = \mathsf{p}_2, \qquad \mathsf{p}_1 + \mathsf{p}_2 = 1 \tag{A.4}$$

that $p(\mathsf{a}) = p(a_1)\mathsf{p}_1 + p(a_2)\mathsf{p}_2$ for any polynomial $p(x)$. Let P be a topological space of functions $\mathbb{R} \to \mathbb{R}$ such that polynomials are dense in it. Then for any $f \in P$ we can define $f(\mathsf{a})$ by the formula

$$f(\mathsf{a}) = f(a_1)\mathsf{p}_1 + f(a_2)\mathsf{p}_2. \tag{A.5}$$

This definition gives continuous algebraic homomorphism.

A.4 (Principal series representations of $SL(2,\mathbb{R})$) We describe a realization of the principal series representations of $SL(2,\mathbb{R})$. The realization is deduced from the realization by left regular representation on the a space of homogeneous function of power $-is-1$ on $\mathbb{R}^2$ described in [23, § 8.3]. We consider now the restriction of homogeneous function not to the unit circle as in [23, Chap. 8, (3.23)] but to the line $x_2 = 1$ in $\mathbb{R}^2$. Then an equivalent unitary representation of $SL(2,\mathbb{R})$ acts on the Hilbert space $L_2(\mathbb{R})$ with the standard Lebesgue measure by the transformations:

$$[\pi_{is}(g)f](x) = \frac{1}{|cx+d|^{1+is}} f\left(\frac{ax+b}{cx+d}\right), \qquad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{A.6}$$

A.5 (Boundedness of the Singular Integral Operator $\mathcal{W}_\sigma$) The kernel of integral operator $\mathcal{W}_\sigma$ (3.24) is singular in four points, which are the intersection of $\widetilde{\mathbb{T}}$ and the light cone with the origin in u. One can easily see

$$\left|\frac{(-\mathsf{u}e_1 e^{e_{12}t}+1)^\sigma}{(-e^{-e_{12}t}e_1\mathsf{u}+1)^{1+\sigma}}\right| = \left|1+\mathsf{u}^2\right|^{1/2} \frac{1}{|t-t_0|} + O(\frac{1}{|t-t_0|^2}).$$

where t_0 is one of four singular points mentioned before for a fixed u and t is a point in its neighborhood. More over the kernel of integral operator $\mathcal{W}_\sigma$ is changing the sign while t crossing the t_0. Thus we can define $\mathcal{W}_\sigma$ in the sense of the principal value as the standard singular integral operator.

Such defined integral operator $\mathcal{W}_\sigma$ becomes a bounded linear operator $L_2(\widetilde{\mathbb{T}}) \to L_2(\widetilde{\mathbb{T}}^\lambda)$, where $\widetilde{\mathbb{T}}^\lambda$ is the circle (A.2) in $\widetilde{\mathbb{R}}^{1,1}$ with center in the origin and the "radius" λ. Moreover the norm of the operator $\lambda^{-2}\mathcal{W}_\sigma$ is uniformly bounded for all λ and thus we can consider it as bounded operator

$$L_2(\widetilde{\mathbb{T}}) \to H_\sigma(\widetilde{\mathbb{D}}),$$

where

$$H_\sigma(\widetilde{\mathbb{D}}) = \{f(\mathsf{u}) \mid D_{\widetilde{\mathbb{D}}} f(\mathsf{u}) = 0,\ \mathsf{u} \in \widetilde{\mathbb{D}},\ |\lambda|^{-2} \int_{\widetilde{\mathbb{T}}^\lambda} |f(\mathsf{u})|^2 \, du < \infty,\ \forall \lambda < 0\}. \tag{A.7}$$

is an analog of the classic Hardy space. Note that $|\lambda|^{-2} \, d\mathsf{u}$ is exactly the invariant measure (3.12) on $\widetilde{\mathbb{D}}$.

One can note the similarity of arising divergency and singularities with the ones arising in quantum field theory. The similarity generated by the same mathematical object in basement: a pseudoeuclidean space with an indefinite metric.

A.6 (Open problems) This paper raises more questions than gives answers. Nevertheless it is useful to state some open problems explicitly.

1. Demonstrate that Cauchy formula (3.25) is an isometry between $F_2(\widetilde{\mathbb{T}})$ and $H_\sigma(\widetilde{\mathbb{D}})$ with suitable norms chosen. This almost follows (up to some constant factor) from its property to intertwine two irreducible representations of $SL(2,\mathbb{R})$.

2. Formula (3.24) contains Szegö type kernel, which is domain dependent. Integral formula (3.25) formulated in terms of analytic kernel. Demonstrate using Stocks theorem that (3.25) is true for other suitable chosen domains.

3. The image of Szegö (or Cauchy) type formulas belong to the kernel of Dirac type operator only if they connected by additional condition (see Proposition 3.9). Descriptive condition for the discrete series can be found in [17, Theorem 6.1]. Formulate a similar condition for principal series representations.

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